Sec 1.6 Exact Eqs / Reducible 2nd order

- An eq of the form* $M\,dx + N\,dy = 0$ $\left(\begin{array}{l} M = M(x,y) \\ N = N(x,y) \end{array}\right)$
is "an <u>exact</u> equation" if $M_y = N_x$ $\left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$

* could be written as $\left\{ M + N\frac{dy}{dx} = 0 \right\}$

★ Can always find an implicit solution $F(x,y) = C$ for an exact ODE.

Why? Idea: (Potential function F)

- Implicit solution existing, $F(x,y) = C$

$\Rightarrow$ total differential $dF = \underbrace{\boxed{\frac{\partial F}{\partial x}}dx + \boxed{\frac{\partial F}{\partial y}}dy}_{M\,dx + N\,dy} = 0$

$M\,dx + N\,dy = 0$

(often "C^2")

- "smooth" functions. $F(x,y)$ <u>will have</u> $F_{xy} = F_{yx}$

$$\underline{M_y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \underline{N_x} \quad \text{(exact)}$$

So ... implicit solution $\Longrightarrow$ exact ODE

In fact, exact equation $\Longrightarrow$ implicit solution.

<u>Ex</u>: Find an implicit solution for

$$\left\{ (\underbrace{6xy - y^3}_{M})dx + (\underbrace{4y + 3x^2 - 3xy^2}_{N})dy = 0 \right\}$$

Check if exact: $\left.\begin{array}{l} M_y = 6x - 3y^2 \\ N_x = 6x - 3y^2 \end{array}\right\}$ equal, so eq is "exact" ✓

we need F such that $\frac{\partial F}{\partial x} = M$, $\frac{\partial F}{\partial y} = N$

$$\frac{\partial F}{\partial x} = 6xy - y^3 \Longrightarrow F = \int (6xy - y^3)dx = 3x^2y - xy^3 + g(y)$$

$F(x,y) = 3x^2y - xy^3 + g(y)$,

need $\frac{\partial F}{\partial y} = N \longrightarrow \cancel{3x^2} - \cancel{3xy^2} + g'(y) = 4y + \cancel{3x^2} - \cancel{3xy^2}$

$$\Longrightarrow g'(y) = 4y$$

$$\Longrightarrow g(y) = 2y^2 + C_1$$

In total, $F(x,y) = 3x^2y - xy^3 + 2y^2 + C_1$

Implicit solution is $\underline{F(x,y) = C}$

$$\boxed{3x^2y - xy^3 + 2y^2 = C}$$

Some 2nd order eqs can be "reduced" into a 1st order eq by setting/subbing $\boxed{v = y'}$

(Note: 2nd order ODE general solutions will have 2 unknown constants)

Two types can have this:

<u>Type "y missing"</u> (x, y', y'' can appear, but no y)

<u>use $v = y'$, $y'' = v'$</u>

(*also assume $x>0$)

<u>Ex:</u> $\{xy'' + y' = x^*\} \longrightarrow \{y'' + \frac{1}{x}y' = 1\}$

with $y' = v$, $y'' = v'$,

eq becomes $\{v' + \frac{1}{x}v = 1\}$ 1st order linear

complement ρ (not q) $= e^{\int P\,dx}$ ~~($v' + \frac{1}{x}v$~~ $v' + Pv = Q$

$P = 1/x$, $Q = 1$

$$\rho = e^{\int 1/x\,dx} = e^{\ln x} = x$$

complement $\rho v = \int \rho Q\,dx + C_1$

$$xv = \int x\,dx + C_1$$

$$\boxed{v = \frac{1}{2}x + \frac{C_1}{x}}$$

since $v = y'$ we now have ODE

$$\left\{ y' = \frac{1}{2}x + \frac{C_1}{x} \right\} \text{ (separable)}$$

. . .

$$\frac{dy}{dx} = \frac{x}{2} + \frac{C_1}{x} \longrightarrow \int dy = \int \left(\frac{x}{2} + \frac{C_1}{x}\right) dx$$

$$y = \frac{x^2}{4} + C_1 \ln|x| + C_2$$

(dropped abs value b/c we are told $x > 0$)

$$\boxed{y = \frac{x^2}{4} + C_1 \ln x + C_2}$$

we would need two data points $\to \left\{ \begin{array}{l} y(a_1) = b_1, \\ y'(a_2) = b_2 \end{array} \right\}$ to solve for particula C_1, C_2 values.

Type "x missing"   Ex: $\left\{ y'' = -25y^{\boxed{*}} \right\}$ $y'' = \frac{d^2y}{dx^2}$

($\boxed{*}$ assume $x, y', y > 0$)

again use $\boxed{y' = v}$ but what to do about y''?

Trick: $y'' = v' = \frac{dv}{dx} = \frac{dv}{dy} \cdot \frac{dy}{dx} = \boxed{v\frac{dv}{dy}}$

so now eq becomes $\left\{ v\frac{dv}{dy} = -25y \right\}$ (separable)

$$\int v\,dv = \int -25y\,dy$$

$$\frac{1}{2}v^2 = -\frac{25}{2}y^2 + C_1$$

$$v^2 = -25y^2 + C_1$$

$$y' = v = + \sqrt{C_1 - 25y^2}$$

we would normally have $\pm\sqrt{\cdots}$ but since we are told to assume $y' > 0$, we just use $+$

$$\left\{ y' = \sqrt{c_1 - 25y^2}, \quad (y, y', x > 0) \right\}$$

separable: $\dfrac{dy}{\sqrt{c_1 - 25y^2}} = dx$

★ trick: $\dfrac{dy}{\sqrt{c_1 - 25y^2}} = \dfrac{dy}{5\sqrt{\frac{c_1}{25} - y^2}} = \dfrac{dy}{5\sqrt{c_1 - y^2}}$

$\longrightarrow \quad \displaystyle\int \frac{dy}{\sqrt{c_1 - y^2}} = \int 5\,dx$

Recall: $\displaystyle\int \frac{dx}{\sqrt{k^2 - x^2}} = \sin^{-1}\left(\frac{x}{k}\right) + C$

$$\sin^{-1}\left(\frac{y}{\sqrt{c_1}}\right) = 5x + c_2$$

($\sqrt{c_1}$ still constant) $\longrightarrow$ $\dfrac{y}{c_1} = \sin(5x + c_2)$

$$\boxed{y = c_1 \sin(5x + c_2)}$$

2nd order eq, so we have 2 constants c_1, c_2.

Again we'd need 2 data points, say $\left\{ \begin{array}{l} y(a_1) = b_1 \\ y'(a_2) = b_2 \end{array} \right\}$ to get a particular solution (by solving for c_1, c_2).